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HUA SYSTEM AND PLURIHARMONICITY FOR SYMMETRIC IRREDUCIBLE SIEGEL DOMAINS OF TYPE II

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Abstract. We consider here a generalization of the Hua system which was proved by Johnson and Korányi to characterize Poisson-Szegö integrals for Siegel domains of tube type. We show that the situation is completely different when dealing with non tube type symmetric irreducible symmetric domains: then all functions which are annihilated by this second order system and satisfy an H^2 type integrability condition are pluriharmonic functions.

1. Introduction

Let $\mathcal{D}$ be a bounded symmetric domain in $\mathbf{C}^m$, and let G be the group of all biholomorphic transformations of $\mathcal{D}$. The aim of this paper is to study $\mathbf{H}$-harmonic functions, where $\mathbf{H}$ is a naturally defined G-invariant real system of second order differential operators on $\mathcal{D}$ which annihilates pluriharmonic functions. The system $\mathbf{H}$ is defined in terms of the Kähler structure of $\mathcal{D}$ and makes sense on every Kählerian manifold.

To define the system $\mathbf{H}$, we recall some basic facts about $\mathcal{D}$. Let $T^{1,0}(\mathcal{D})$ be the holomorphic tangent bundle of $\mathcal{D}$. The Riemannian connection ∇ induced by the Bergman metric on $\mathcal{D}$ preserves $T^{1,0}(\mathcal{D})$ and so does the curvature tensor. For Z, W two complex vector fields we denote by $R(Z,W) = \nabla_Z\nabla_W - \nabla_W\nabla_Z - \nabla_{[Z,W]}$ the curvature tensor restricted to $T^{1,0}(\mathcal{D})$. Let f be a smooth function on $\mathcal{D}$ and let

$$\Delta(Z,W)f = (Z\overline{W} - \nabla_Z\overline{W})f = (\overline{W}Z - \nabla_{\overline{W}}Z)f\,. \tag{1}$$

Then $\Delta(Z,W)$ may be seen as a second order operator which annihilates both holomorphic and antiholomorphic functions, and consequently, the pluriharmonic functions. Conversely, if all $\Delta(Z,W)$ annihilate f, then f is pluriharmonic. Indeed, we have $\Delta(\partial_{z_j}, \partial_{z_k}) = \partial_{z_j}\partial_{\overline{z_k}}$.

Let $(\ ,\ )$ be the canonical Hermitian product in $T^{1,0}(\mathcal{D})$. Fixing a smooth function f, we use $(\ ,\ )$ to define a smooth section Δ_f of the bundle of endomorphisms of

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$T^{1,0}(\mathcal{D})$:

$$(\Delta f \cdot Z, W) = \Delta(W, Z) f \,, \tag{2}$$

where Z, W are holomorphic vector fields. Then we define $\mathbf{H}f$ as another smooth section of the bundle of endomorphisms of $T^{1,0}(\mathcal{D})$ by

$$(\mathbf{H}f \cdot Z, W) = \mathrm{Tr}(R(\overline{Z}, W)^* \Delta_f) = \mathrm{Tr}(R(\overline{W}, Z)\Delta_f) \,. \tag{3}$$

To compute explicitly $\mathbf{H}f$, we may take an orthonormal frame of sections of $T^{1,0}(\mathcal{D})$, which we denote $E_1, E_2, \cdots, E_m$. Then

$$\mathbf{H}f = \sum_{j,k} (\Delta(E_j, E_k) f) R(\overline{E}_j, E_k) \,. \tag{4}$$

The system $\mathbf{H}$ is, of course, a contraction of the tensor field Δ_f. It is invariant with respect to biholomorphisms, which means that

$$\mathbf{H}(f \circ \Psi) = \Psi_*^{-1}[(\mathbf{H}f) \circ \Psi]\Psi_* \tag{5}$$

for every biholomorphic transformation Ψ of $\mathcal{D}$, Ψ_* being its differential.

By definition, $\mathbf{H}$-harmonic functions are functions which are annihilated by $\mathbf{H}$. We will consider here symmetric Siegel domains, for which these notions are well defined since they are holomorphically equivalent to bounded domains. When $\mathcal{D}$ is a symmetric Siegel domain of *tube type*, (3) is equivalent to the classical Hua system. This system is known to characterize the Poisson-Szegö integrals (see [FK] and [JK]). This means that a function on $\mathcal{D}$ is $\mathbf{H}$-harmonic if, and only if, it is the Poisson-Szegö integral of a hyperfunction on the Shilov boundary. Originally, the curvature tensor was not explicit in the Hua system. For classical domains, the system has been defined by L. K. Hua as a “quantization” of the equation defining the Shilov boundary (see [Hu] and [BV]). L. K. Hua proved that the system annihilates Poisson-Szegö integrals. Then the system was extended by K. Johnson and A. Korányi [JK] to all symmetric tube type domains and was written down in terms of the enveloping algebra of the semi-simple Lie group of automorphisms of the domain. K. Johnson and A. Korányi proved not only that for all tube domains the system annihilates the Poissonn-Szegö kernel, but also that the $\mathbf{H}$-harmonic functions are Poisson-Szegö integrals. Rewriting Johnson–Korányi formula $C(\partial, \bar{\partial})$ in terms of the curvature tensor, as suggested by Nolan Wallach, one obtains the same system as above. It is why we call $\mathbf{H}$ the Hua-Wallach system.

Notice that (4) and (5) have a perfect sense on any Kählerian manifold, and, for general Siegel domains, the system (4) has been already studied in [DHP]. In particular, for non tube symmetric Siegel both (4) and Johnson–Koranyi formula $C(\partial, \bar{\partial})$ take the same form. In the work of N. Berline and M. Vergne [BV] it is observed that $C(\partial, \bar{\partial})$ does not annihilate Poisson - Szegö integrals, and the problem of describing $C(\partial, \bar{\partial})$–harmonic functions is risen. Here we are going to answer their question.

Main Theorem. *Let $\mathcal{D}$ be a symmetric irreducible Siegel domain of type II, and let F be an $\mathbf{H}$-harmonic function on $\mathcal{D}$ which satisfies the growth condition*

$$(H^2) \qquad \sup_{z\in\mathcal{D}} \int_{N(\Phi)} |F(uz)|^2 du < \infty,$$

where $N(\Phi)$ is a nilpotent subgroup of S whose action is parallel to the Shilov boundary. Then F is pluriharmonic.

This is in a striking contrast to the case when $\mathcal{D}$ is a symmetric tube domain. It requires some comments.

The Poisson-Szegö integrals on type II domains have been characterized by N. Berline and M. Vergne [BV] as zeros of a G-invariant system which, in general, is of the third order. It is obtained by "quantization" of the Shilov boundary equations. They also prove that for domains over the cone of hermitian positive definite matrices one can use a second order system, Δ_Z, to characterize Poisson-Szegö integrals. This system appears already in the book by Hua [Hu]. It is obtained from $C(\partial, \bar{\partial})$ by a projection that eliminates a part of the equations.

All this shows that the system $\mathbf{H}$ does not seem to be canonical in any sense, although it is defined with the aid of the curvature tensor, certainly an important invariant, the geometric meaning of the system being still unclear. Our present work suggests that it would be interesting to understand second order systems of operators on symmetric Siegel domains which are invariant under the full group of biholomorphisms.

In the proof of the main theorem, we use heavily the theory of harmonic functions with respect to subelliptic operators on solvable Lie groups [R], [D], [DH], [DHP]. To do this, we identify the domain $\mathcal{D}$ with a solvable Lie group $S \subset G$ that acts simply transitively on $\mathcal{D}$. We then use a special orthonormal frame of S-invariant vector fields, $E_1, E_2, \cdots, E_m$, to compute the operator $\mathbf{H}$ by the formula (4). In fact, we only consider the left-invariant second order elliptic operators built out of the diagonal of $\mathbf{H}$,

$$(6) \qquad \mathbf{H}_j f = (\mathbf{H}f \cdot E_j, E_j).$$

Elliptic operators which are linear combinations of operators $\mathbf{H}_j$ play the main role in our argument, and in particular the Laplace-Beltrami operator Δ, which is the trace of $\mathbf{H}$. We represent $\mathbf{H}$-harmonic functions as various Poisson integrals, and we use properties of these representations.

Linear combinations of the operators $\Delta(E_j, E_k)$ have already been used to characterize pluriharmonic functions (see [DHMP]). We should emphasize that the systems under study here are different from those of [DHMP], and the proofs require new ideas. Since a part of the construction is the same in the two papers, we try to simplify the presentation for the reader's convenience.

Our growth assumption (H^2) is made mainly for technical reasons, L^2 harmonic analysis being the easiest. We hope to be able to obtain similar conclusions for bounded functions, and perhaps even for larger classes of functions. This requires,

however, a somewhat more delicate technic. In fact, it is not clear that the conclusion requires any growth condition at the boundary, and one may conjecture that only growth conditions at infinity are necessary to insure pluriharmonicity for **H**–harmonic functions on symmetric irreducible Siegel domains of type II. On the other hand, for tube type domains, one may conjecture that growth conditions on derivatives at the boundary insure pluriharmonicity for **H**-harmonic functions as it is the case in the unit ball ([BBG]) for Δ-harmonic functions.

Finally, let us remark that, if we do not insist on invariance properties of the systems considered, then it is always possible to characterize pluriharmonic functions, among the functions which are harmonic with respect to the Laplace-Beltrami operator Δ, as those which are annihilated by a single second order operator L (without any growth condition). Indeed, a classical theorem of Forelli (see Rudin's book [Ru]) asserts that every smooth function in the unit ball which is annihilated by the operator $\sum z_j \overline{z_k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}}$ is pluriharmonic in the ball. So L can be taken as this operator suitably translated, so that a function which is annihilated by L is pluriharmonic in the neihborhood of a point. Then the real-analyticity of the function, which follows from the fact that it is Δ–harmonic, insures its pluriharmonicity everywhere.

In view of Forelli's Theorem, it is not so much the small number of operators in the system used to characterize pluriharmonic functions than the strong invariance properties of the system itself which are relevant. In this context, the present paper can be viewed as a complement to [DHP] and[DHMP].

2. Hua-Wallach systems

2.1. **General Hua-Wallach systems.** In this subsection, $\mathcal{D}$ is a general domain in $\mathbb{C}^m$ which is holomorphically equivalent to a bounded domain. We recall here the properties of the Kählerian structure related to the Bergman matric as well as some elementary facts about the Hua-Wallach system which we will use later. The reader may refer to [He] and [KN] for more details on the prerequisites.

Let T be the tangent bundle for the complex domain $\mathcal{D}$, and let $T^{\mathbf{C}}$ be the complexified tangent bundle. The complex structure $\mathcal{J}$ and the Bergman metric g are extended from T to $T^{\mathbf{C}}$ by complex linearity. Let $T^{1,0}$ and $T^{0,1}$ be the eigenspaces of $\mathcal{J}$ such that $\mathcal{J}|_{T^{1,0}} = i\mathrm{Id}$, $\mathcal{J}|_{T^{0,1}} = -i\mathrm{Id}$. We have

$$T^{\mathbf{C}} = T^{1,0} \oplus T^{0,1} .$$

The conjugation operator exchanges $T^{1,0}$ and $T^{0,1}$.

The spaces of smooth sections of T, $T^{\mathbf{C}}$, $T^{1,0}$ will be denoted $\Gamma(T)$, $\Gamma(T^{\mathbf{C}})$, $\Gamma(T^{1,0})$, respectively. Smooth sections of $T^{1,0}$ are called holomorphic vector fields.

The Riemannian connection ∇ is also extended from $\Gamma(T)$ to $\Gamma(T^{\mathbf{C}})$ by complex linearity and, since ∇ is defined by a Kählerian structure, it commutes with $\mathcal{J}$. An immediate consequence is that $\nabla_Z W$ belongs to $\Gamma(T^{1,0})$ (respectively $\Gamma(T^{0,1})$) whenever $W \in \Gamma(T^{1,0})$ (respectively $\Gamma(T^{0,1})$). Moreover, for every couple $U, V \in$

$\Gamma(T^{\mathbf{C}})$, we have that $\overline{\nabla_U V} = \nabla_{\overline{U}}\overline{V}$ and

$$[\overline{U}, V] = \nabla_{\overline{U}} V - \nabla_V \overline{U}, . \tag{7}$$

Therefore, for Z, W holomorphic vector fields,

$$\nabla_{\overline{W}} Z = \pi_{(1,0)}([\overline{W}, Z]), \tag{8}$$

where $\pi_{(1,0)}$ denotes the projection from $T^{\mathbf{C}}$ onto $T^{1,0}$. The curvature tensor

$$R(U,V) = \nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U,V]}$$

preserves also $T^{1,0}$ and $R(\overline{U}, \overline{V})\overline{Z} = \overline{R(U,V)Z}$. The restriction of $R(U,V)$ to $T^{1,0}$ is also denoted by $R(U,V)$. On $T^{1,0}$ the Hermitian scalar product arising from the Bergman metric is denoted by

$$(Z,W) = \frac{1}{2} g(Z, \overline{W}).$$

For U, V, Z, W holomorphic vector fields, we have

$$(R(\overline{V}, U)Z, W) = (R(\overline{W}, Z)\, U, V) = (U, R(\overline{Z}, W)\, V)\,. \tag{9}$$

In particular,

$$R(\overline{W}, Z) = R(\overline{Z}, W)^*\,. \tag{10}$$

Let us now go back to the definitions given in the introduction. The identity $Z\overline{W} - \nabla_Z \overline{W} = \overline{W} Z - \nabla_{\overline{W}} Z$ is a direct consequence of (7). The fact that all $\Delta(Z,W)$ annihilate pluriharmonic functions follows from (8) as well as from the identity $\Delta(\partial_{z_j}, \partial_{z_k}) = \partial_{z_j} \partial_{\overline{z_k}}$. Moreover,

$$\Delta(\phi Z, \psi W) f = \phi\, \overline{\psi}\, \Delta(Z,W) f\,.$$

which means that Δ_f is a tensor field. The equality in (3) comes from (7), while one proves (4) using (9) for $(R(\overline{W}, Z) \cdot E_k, E_j)$.

Let us now show invariance of $\mathbf{H}_{\mathcal{D}}$ with respect to biholomorphisms. Let Ψ be a biholomorphism from $\mathcal{D}$ onto $\mathcal{D}'$, and Ψ_* the holomorphic differential of Ψ which maps $T^{1,0}_{\mathcal{D}}$ onto $T^{1,0}_{\mathcal{D}'}$. All tensor fields are transported by Ψ, including, of course, the Riemannian structure and the curvature tensor. Thus

$$R_{\mathcal{D}}(\overline{W}, Z) = \Psi_*^{-1} R_{\mathcal{D}'}(\overline{\Psi_* W}, \Psi_* Z) \Psi_*\,.$$

Moreover, for a smooth function f on $\mathcal{D}'$ and $g = f \circ \Psi$, we have $\Delta_g = \Psi_*^{-1} \Delta_f \Psi_*$. So

$$(\mathbf{H}_{\mathcal{D}}\, g \cdot Z, W) = \operatorname{Tr}\left(R_{\mathcal{D}'}(\overline{\Psi_* W}, \Psi_* Z)\Delta_f\right)$$

which implies

$$\mathbf{H}_{\mathcal{D}}\, g = \Psi_*^{-1} (\mathbf{H}_{\mathcal{D}'}\, f) \Psi_*\,.$$

Finally, let us remark that, from formulas (1), (3) and (10), it follows that $\Delta_{\overline{f}} = (\Delta_f)^*$, and $\mathbf{H}\overline{f} = (\mathbf{H} f)^*$. So, to study $\mathbf{H}$-harmonic functions, it is sufficient to consider functions which are real-valued.

We want now to compute explicitly the Hua-Wallach operator for symmetric irreducible Siegel domains. To do it, we will use Formula (4) for a particular orthonormal basis $E_1, ..., E_m$.

2.2. **Preliminaries on irreducible symmetric cones.** Let Ω be an irreducible symmetric cone in an Euclidean space. Our aim is to describe precisely the solvable group that acts simply transitively on Ω. The group will be used in the construction of the orthonormal basis. We do it all in terms of Jordan algebras, and we refer to the book of Faraut and Korányi [FK] for these prerequisites, introducing here only the notations and principal results that will be needed later.

A finite dimensional algebra V with a scalar product $\langle \cdot, \cdot \rangle$ is an Euclidean Jordan algebra if for all elements x, y and z in V

$$xy = yx \qquad x(x^2 y) = x^2(xy) \qquad \langle xy, z\rangle = \langle y, xz\rangle.$$

We denote by $L(x)$ the self-adjoint endomorphism of V given by the multiplication by x, i.e. $L(x)y = xy$.

For an irreducible symmetric cone Ω contained in a linear space V of same dimension, the space V can be made a simple real Euclidean Jordan algebra with unit element e, so that

$$\Omega = \text{ int}\, \{x^2 : \ x \in V\}.$$

Let G be the connected component of the group of all transformations in $GL(V)$ which leave Ω invariant, and let $\mathcal{G}$ be its Lie algebra. Then $\mathcal{G}$ is a subspace of the space of endomorphisms of V which contains all $L(x)$ for all $x \in V$, as well as all $x \square y$ for $x, y \in V$, where $x \square y = L(xy) + [L(x), L(y)]$ (see [FK] for these properties).

We fix a Jordan frame $\{c_1, \ldots, c_r\}$ in V, that is, a complete system of orthogonal primitive idempotents:

$$c_i^2 = c_i, \qquad c_i c_j = 0 \quad \text{if } i \neq j, \qquad c_1 + ... + c_r = e$$

and none of the $c_1, ..., c_r$ is a sum of two non-zero idempotents. Let us recall that the length r is independent of the choice of the Jordan frame. It is called the rank of V. To have an example in mind, one may think of the space V of the symmetric $r \times r$ matrices endowed with the symmetrized product of matrices $\frac{1}{2}(xy + yx)$. Then the corresponding cone is the set of symmetric positive definite $r \times r$ matrices, the set of diagonal matrices with all entries equal to 0 except for one equal to 1 being a Jordan frame.

The Peirce decomposition of V related to the Jordan frame $\{c_1, \ldots, c_r\}$ ([FK], Theorem IV.2.1) may be written as

$$V = \bigoplus_{1 \leq i \leq j \leq r} V_{ij}\,. \tag{11}$$

It is given by the common diagonalization of the self-adjoint endomorphims $L(c_j)$ with respect to their only eigenvalues 0, $\frac{1}{2}$, 1. In particular $V_{jj} = \mathbb{R} c_j$ is the eigenspace of $L(c_j)$ related to 1, and, for $i < j$, V_{ij} is the intersection of the

eigenspaces of $L(c_i)$ and $L(c_j)$ related to $\frac{1}{2}$. All V_{ij}, for $i < j$, have the same dimension d.

For each $i < j$, we fix once for all an orthonormal basis of V_{ij}, which we note $\{e_{ij}^\alpha\}$, with $1 \leq \alpha \leq d$. To simplify the notation, we write $e_{ii}^\alpha = c_i$ (α taking only the value 1). Then the system $\{e_{ij}^\alpha\}$, for $i \leq j$ and $1 \leq \alpha \leq \dim V_{ij}$, is an orthonormal basis of V.

Let us denote by $\mathcal{A}$ the abelian subalgebra of $\mathcal{G}$ consisting of elements $H = L(a)$, where

$$a = \sum_{j=1}^r a_j c_j \in \bigoplus_i V_{ii}.$$

We set λ_j the linear form on $\mathcal{A}$ given by $\lambda_j(H) = a_j$. It is clear that the Peirce decomposition gives also a simultaneous diagonalization of all $H \in \mathcal{A}$, namely

$$Hx = L(a)x = \frac{\lambda_i(H) + \lambda_j(H)}{2} x \qquad\qquad x \in V_{ij}\,. \tag{12}$$

Let $A = \exp \mathcal{A}$. Then A is an abelian group, and this is the Abelian group in the Iwasawa decomposition of G. We now describe the nilpotent part N_0. Its Lie algebra $\mathcal{N}_0$ is the space of elements $X \in \mathcal{G}$ such that, for all $i \leq j$,

$$XV_{ij} \subset \bigoplus_{k \geq l\ ;\ (k,l) > (i,j)} V_{kl},$$

where the pairs ordered lexicographically. Once $\mathcal{N}_0$ is defined, we define $\mathcal{S}_0$ as the direct sum $\mathcal{N}_0 \oplus \mathcal{A}$. The groups S_0 and N_0 are then obtained by taking the exponentials. It follows from the definition of $\mathcal{N}_0$ that the matrices of elements of $\mathcal{S}_0$ and S_0, in the orthonormal basis $\{e_{ij}^\alpha\}$, are upper-triangular.

The solvable group S_0 acts simply transitively on Ω. This may be found in [FK] Chapter VI, as well as the precise description of $\mathcal{N}_0$ which will be needed later. One has

$$\mathcal{N}_0 = \bigoplus_{i<j\leq r} \mathcal{N}_{ij},$$

where

$$\mathcal{N}_{ij} = \{z \square c_i \ : z \ \in V_{ij}\}.$$

This decomposition corresponds to a diagonalization of the adjoint action of $\mathcal{A}$ since

$$[H, X] = \frac{\lambda_j(H) - \lambda_i(H)}{2} X, \quad X \in \mathcal{N}_{ij}\,. \tag{13}$$

Finally, let $V^{\mathbf{C}} = V + iV$ be the complexification of V. We extend the action of G to $V^{\mathbf{C}}$ in the obvious way.

2.3. **Preliminaries on irreducible symmetric Siegel domains of type II.** We consider the Siegel domain defined by an irreducible symmetric cone Ω and an additional complex vector space $\mathcal{Z}$ together with a Hermitian symmetric bilinear mapping

$$\Phi : \mathcal{Z} \times \mathcal{Z} \to V^{\mathbf{C}},$$

such that

$$\Phi(\zeta,\zeta) \in \overline{\Omega}, \quad \zeta \in \mathcal{Z}\,,$$
$$\Phi(\zeta,\zeta) = 0 \ \text{ implies } \zeta = 0.$$

The Siegel domain associated with these data is defined as

$$\mathcal{D} = \{(\zeta, z) \in \mathcal{Z} \times V^{\mathbf{C}} : \ \Im z - \Phi(\zeta,\zeta) \in \Omega\}. \tag{14}$$

It is called of tube type if $\mathcal{Z}$ is reduced to $\{0\}$. Otherwise, it is called of type II. There is a representation $\sigma : S_0 \ni s \mapsto \sigma(s) \in GL(\mathcal{Z})$ such that

$$s\Phi(\zeta, w) = \Phi(\sigma(s)\zeta, \sigma(s)w)\,, \tag{15}$$

and such that all automorphisms $\sigma(s)$, for $s \in A$, admit a joint diagonalization (see [KW]). To reduce notations, we shall as well denote by σ the corresponding representation of the algebra $\mathcal{S}_0$. For $X \in \mathcal{S}_0$, (15) implies that

$$X\Phi(\zeta, w) = \Phi(\sigma(X)\zeta, w) + \Phi(\zeta, \sigma(X)w)\,. \tag{16}$$

As an easy consequence, one can prove that the only possible eigenvalues for $\sigma(H)$, with $H \in \mathcal{A}$, are $\lambda_j(H)/2$, for $j = 1, ..., r$. So we may write

$$\mathcal{Z} = \bigoplus_{j=1}^{r} \mathcal{Z}_j$$

with the property that

$$\sigma(H)\zeta = \frac{\lambda_j(H)}{2}\zeta, \qquad \zeta \in \mathcal{Z}_j. \tag{17}$$

Moreover, all the spaces $\mathcal{Z}_j$ have the same dimension[1]. A proof of these two facts may be found in [DHMP]. We call χ the dimension of $\mathcal{Z}_j$ for $j = 1, ..., r$. Let us remark, using (16) and (17), that for $\zeta, w \in \mathcal{Z}_j$, we have $L(c_j)\Phi(\zeta, w) = \Phi(\zeta, w)$. Therefore, $\Phi(\zeta, w) = Q_j(\zeta, w)c_j$, for $\zeta, w \in \mathcal{Z}_j$. Moreover, $\langle c_j, \Phi(\zeta,\zeta)\rangle > 0$ for $\zeta \in \mathcal{Z}_j$ and so the Hermitian form Q_j is positive definite on $\mathcal{Z}_j$.

The representation σ allows to consider S_0 as a group of holomorphic automorphisms of $\mathcal{D}$. More generally, the elements $\zeta \in \mathcal{Z}$, $x \in V$ and $s \in S_0$ act on $\mathcal{D}$ in

[1] In fact, the present study generalizes to all homogeneous Siegel domains related to irreducible symmetric cones for which this last property is satisfied.

the following way:

$$\begin{aligned} \zeta\cdot(w,z) &= (\zeta+w, z+2i\Phi(w,\zeta)+i\Phi(\zeta,\zeta))\,,\\ x\cdot(w,z) &= (w, z+x)\,,\\ s\cdot(w,z) &= (\sigma(s)w, sz)\,. \end{aligned} \tag{18}$$

We call $N(\Phi)$ the group corresponding to the first two actions, that is $N(\Phi)=\mathcal{Z}\times V$ with the product

$$(\zeta,x)(\zeta',x') = (\zeta+\zeta', x+x'+2\Im\,\Phi(\zeta,\zeta')). \tag{19}$$

All three actions generate a solvable Lie group

$$S = N(\Phi)S_0 = N(\Phi)N_0A = NA,$$

which identifies with a group of holomorphic automorphisms acting simply transitively on $\mathcal{D}$. The group $N(\Phi)$, that is two-step nilpotent, is a normal subgroup of S. The Lie algebra $\mathcal{S}$ of S admits the decomposition

$$\mathcal{S} = \mathcal{N}(\Phi)\oplus\mathcal{S}_0 = \left(\bigoplus_{j=1}^{r}\mathcal{Z}_j\right)\oplus\left(\bigoplus_{i\le j}V_{ij}\right)\oplus\left(\bigoplus_{i<j}\mathcal{N}_{ij}\right)\oplus\mathcal{A}. \tag{20}$$

Moreover, by (12), (13) and (17), one knows the adjoint action of elements $H\in\mathcal{A}$:

$$\begin{aligned} [H,X] &= \frac{\lambda_j(H)}{2}X \qquad\quad \text{for}\quad X\in\mathcal{Z}_j,\\ [H,X] &= \frac{\lambda_i(H)+\lambda_j(H)}{2}X \quad \text{for}\quad X\in V_{ij},\\ [H,X] &= \frac{\lambda_j(H)-\lambda_i(H)}{2}X \quad \text{for}\quad X\in\mathcal{N}_{ij}. \end{aligned} \tag{21}$$

Since S acts simply transitively on the domain $\mathcal{D}$, we may identify S and $\mathcal{D}$. More precisely, we define

$$\theta: S\ni s\mapsto\theta(s) = s\cdot\mathbf{e}\in\mathcal{D}, \tag{22}$$

where $\mathbf{e}$ is the point $(0,ie)$ in $\mathcal{D}$. The Lie algebra $\mathcal{S}$ is then identified with the tangent space of $\mathcal{D}$ at $\mathbf{e}$ using the differential $d\theta_e$. We identify e with the unit element of S. We then transport both the Bergman metric g and the complex structure $\mathcal{J}$ from $\mathcal{D}$ to S, where they become left-invariant tensor fields on S. We still write $\mathcal{J}$ for the complex structure on S. Moreover, the complexified tangent space $T_{\mathbf{e}}^{\mathbf{C}}$ is identified with the complexification of $\mathcal{S}$, which we denote by $\mathcal{S}^{\mathbf{C}}$. The decomposition $T_{\mathbf{e}}^{\mathbf{C}} = T_{\mathbf{e}}^{(1,0)}\oplus T_{\mathbf{e}}^{(0,1)}$ is transported into

$$\mathcal{S}^{\mathbf{C}} = \mathcal{Q}\oplus\mathcal{P}\,. \tag{23}$$

Elements of $\mathcal{S}^{\mathbf{C}}$ are identified with left invariant vector fields on S, and are called left invariant holomorphic vector fields when they belong to $\mathcal{Q}$. The conjugation operator exchanges $\mathcal{Q}$ and $\mathcal{P}$, while the transported operator $\mathcal{J}$ coincides with $i\mathrm{Id}$ on $\mathcal{Q}$, and to $-i\mathrm{Id}$ on $\mathcal{P}$. The Kählerian metric given by the Bergman metric can be

seen as a Hermitian form on $\mathcal{Q}$, and orthonormality for left invariant holomorphic vector fields means orthonormality for the corresponding elements in $\mathcal{Q}$.

Now, let us construct an orthonormal basis of left invariant holomorphic vector fields. We first build a basis in $\mathcal{S}$. To do this, we use the decomposition given in (20) and give a basis for each block.

We have already fixed an orthonormal basis $\{e_{jk}^\alpha\}$ in V corresponding to the Peirce decomposition chosen. For $j<k$ and $1\le\alpha\le d$, we define $X_{jk}^\alpha\in V_{jk}$ and $Y_{jk}^\alpha\in\mathcal{N}_{jk}$ as the left-invariant vector fields on S corresponding to e_{jk}^α and $2e_{jk}^\alpha\,\square c_j$, respectively. For each j we define X_j and H_j as the left-invariant vector fields on S corresponding to $c_j\in V_{jj}$ and $L(c_j)\in\mathcal{A}$, respectively. It remains to choose a basis of each $\mathcal{Z}_j$. We choose for $e_{j\alpha}$ an orthonormal basis of $\mathcal{Z}_j$ related to $4Q_j$, where Q_j is the quadratic form defined above. For $z_{j\alpha}=x_{j\alpha}+iy_{j\alpha}$ the corresponding coordinates, we define $\mathcal{X}_j^\alpha,\mathcal{Y}_j^\alpha$ as the left-invariant vector fields on S which coincide with $\partial_{x_{j\alpha}}$ and $\partial_{y_{j\alpha}}$ at $\mathbf{e}$.

Finally, we define

$$Z_j=X_j-iH_j,\qquad Z_{jk}^\alpha=X_{jk}^\alpha-iY_{jk}^\alpha,\qquad \mathcal{Z}_j^\alpha=\mathcal{X}_j^\alpha-i\mathcal{Y}_j^\alpha\,.$$

We can now state the following lemma.

Lemma 2.1. *The left invariant vector fields Z_j, for $j=1,\cdots,r$, Z_{jk}^α, for $j<k\le r$ and $\alpha=1,\cdots,d$, and $\mathcal{Z}_j^\alpha$ for $j=1,\cdots,r$ and $\alpha=1,\cdots,\chi$, constitute an orthonormal basis of holomorphic left invariant vector fields.*

Proof. This lemma is already contained in [DHMP], to which we refer for details. To prove that Z_j, Z_{jk}^α, and $\mathcal{Z}_j^\alpha$ are holomorphic vector fields, it is sufficient to prove that

$$\mathcal{J}(X_j)=H_j,\qquad \mathcal{J}(X_{jk}^\alpha)=Y_{jk}^\alpha,\qquad \mathcal{J}(\mathcal{X}_j^\alpha)=\mathcal{Y}_j^\alpha\,.$$

To do this, we compute the image of the vector fields $X_j,H_j,X_{jk}^\alpha,Y_{ij}^\alpha,\mathcal{X}_j^\alpha$, and $\mathcal{Y}_j^\alpha$ by the differential $d\theta_e$. We find the following tangent vectors at $\mathbf{e}$: $\partial_{x_{jj}}$, $\partial_{y_{jj}}$, $\partial_{x_{jk}^\alpha}$, $\partial_{y_{jk}^\alpha}$, $\partial_{x_{j\alpha}}$, and $\partial_{y_{j\alpha}}$. Here the coordinates that we have used in $\mathcal{Z}\times V^{\mathbf{C}}$ are given by

$$(\zeta,z)=\Big(\sum_{j,\alpha}(x_{j\alpha}+iy_{j\alpha})e_{j\alpha},\sum_{i\le j,\alpha}(x_{ij}^\alpha+iy_{ij}^\alpha)e_{ij}^\alpha\Big).$$

The assertion follows at once, using the complex structure in $\mathcal{Z}\times V^{\mathbf{C}}$.

To show orthonormality, it is possible to use Koszul's formula which allows to get the Bergman metric from the adjoint action. This is done in [DHMP], Lemma (1.18). □

2.4. **Hua-Wallach systems for irreducible symmetric Siegel domains.** We now compute the operator $\mathbf{H}$ in the orthonormal basis that we have built in the previous subsection. In fact, it is enough to compute the following operators, called *strongly diagonal HW* operators, and defined by

$$\mathbf{H}_jf=(\mathbf{H}f\cdot Z_j,Z_j)\ ,\qquad j=1,\cdots,r.$$

We have the following proposition.

Proposition 2.2. *The strongly diagonal HW operators* $\mathbf{H}_j$ *are*

$$\mathbf{H}_j = \sum_\alpha \mathcal{L}_j^\alpha + 2\Delta_j + \sum_{k<j}\sum_\alpha \Delta_{kj}^\alpha + \sum_{l>j}\sum_\alpha \Delta_{jl}^\alpha\,, \tag{24}$$

where

$$\begin{aligned} \Delta_j &= X_j^2 + H_j^2 - H_j \\ \mathcal{L}_j^\alpha &= (\mathcal{X}_j^\alpha)^2 + (\mathcal{Y}_j^\alpha)^2 - H_j \\ \Delta_{ij}^\alpha &= (X_{ij}^\alpha)^2 + (Y_{ij}^\alpha)^2 - H_j\,. \end{aligned} \tag{25}$$

Proof. We first compute the curvature tensor $R(\overline{Z}, Z)$, with $Z = Z_1, \cdots, Z_r$. From (8), we know that, for Z, W in $\mathcal{Q}$,

$$\nabla_{\overline{Z}} W = \pi_{\mathcal{Q}}([\overline{Z}, W]) = \pi_{\mathcal{Q}}([\overline{Z}, (W + \overline{W})]), \tag{26}$$

where $\pi_{\mathcal{Q}}$ denotes the projection from $\mathcal{S}^{\mathbf{C}}$ onto $\mathcal{Q}$. We claim that

Lemma 2.3. *The following identities hold:*

$$\begin{aligned} \nabla_{\overline{Z}_j} Z_k =& i\delta_{jk} Z_j \\ \nabla_{\overline{Z}_j} Z_{kl}^\alpha =& \frac{i}{2}(\delta_{lj} Z_{kj}^\alpha + \delta_{kj} Z_{jl}^\alpha) \quad \text{if} \quad k<l \\ \nabla_{\overline{Z}_j} \mathcal{Z}_k^\alpha =& \frac{i}{2}\delta_{jk}\mathcal{Z}_j^\alpha. \end{aligned}$$

Proof. In the computation, we have seen that we may replace the three left hand sides of the formulas above by $2\pi_{\mathcal{Q}}([\overline{Z}_j, X_k])$, $2\pi_{\mathcal{Q}}([\overline{Z}_j, X_{kl}^\alpha])$ and $2\pi_{\mathcal{Q}}([\overline{Z}_j, \mathcal{X}_k^\alpha])$, respectively. Moreover, if we replace $\overline{Z}_j$ by iH_j in these three expressions, we obtain the right hand sides, by virtue of (21). Thus the lemma follows, once we prove that all brackets $[\overline{Z}_j, X_k]$, $[\overline{Z}_j, X_{kl}^\alpha]$, and $[\overline{Z}_j, \mathcal{X}_k^\alpha]$ vanish. This last fact follows from a standard argument. One proves that each of these vector fields is annihilated by all endomorphisms $\mathrm{ad}H - \lambda(H)\mathrm{Id}$, with $H \in \mathcal{A}$, for a value $\lambda(H)$ that is not an eigenvalue of $\mathrm{ad}H$ for some H. So it vanishes. $\square$

Let us go on with the proof of the proposition. It is easy to deduce the action of ∇_Z on $\mathcal{Q}$ from the one of $\nabla_{\overline{Z}}$. Indeed, since the action of S preserves the Hermitian scalar product, and since Z is left-invariant,

$$0 = Z \cdot (U, V) = (\nabla_Z U, V) + (U, \nabla_{\overline{Z}} V)$$

for any couple U, V of left-invariant holomorphic vector fields. So the endomorphism of $\mathcal{Q}$ defined by ∇_Z is the opposite of the adjoint endomorphism defined by $\nabla_{\overline{Z}}$. It follows from the matrix representation given in the lemma that they are equal, and they commute. So, for $U \in \mathcal{Q}$,

$$R(\overline{Z_j}, Z_j)U = -\nabla_{[\overline{Z}_j, Z_j]} U = -2i\nabla_{\overline{Z_j}} U$$

since $[\overline{Z_j}, Z_j] = 2iX_j = i(Z_j + \overline{Z_j})$. Using again Lemma 2.3 and the expression of $\mathbf{H}f$ given in (4), we see that

$$\mathbf{H}_j = \sum_\alpha \Delta(\mathcal{Z}_j^\alpha, \mathcal{Z}_j^\alpha) + 2\Delta(Z_j, Z_j) + \sum_{k<j}\sum_\alpha \Delta(Z_{kj}^\alpha, Z_{kj}^\alpha) + \sum_{l>j}\sum_\alpha \Delta(Z_{jl}^\alpha, Z_{jl}^\alpha)\,.$$

We refer to [DHMP] for the computation of $\Delta(\mathcal{Z}_j^\alpha, \mathcal{Z}_j^\alpha)$, $\Delta(Z_j, Z_j)$, and $\Delta(Z_{kj}^\alpha, Z_{kj}^\alpha)$. □

We also refer to [DHMP] for the computation of the Laplace-Beltrami operator Δ,

$$\Delta = \sum_j \Delta_j + \sum_{k<j}\sum_\alpha \Delta_{kj}^\alpha + \sum_{j,\alpha} \mathcal{L}_j^\alpha. \tag{27}$$

It is proved in [DHP] that the Laplace-Beltrami operator is the trace of the operator $\mathbf{H}$.

All results, up to now, are also valid for the tube domain $T_\Omega = V + i\Omega$, which identifies with the subgroup VS_0 of the group S and appears as a particular case. Left invariant differential operators act from the right. Therefore, we can identify left invariant differential operators on the tube domain with left invariant differential operators on the domain $\mathcal{D}$ itself. We add a subscript or superscript for such operators coming from the tube domain, and define $\mathbf{H}_j^T$, $j = 1, \cdots, r$, and Δ_T as the operators coming from the strongly diagonal HW operators for the tube domain and the Laplace-Beltrami operator, respectively. Then, we have the following corollary, the proof of which is immediate:

Corollary 2.4. *The following identities hold:*

$$\mathbf{H}_j^T = 2\Delta_j + \sum_{k<j}\sum_\alpha \Delta_{kj}^\alpha + \sum_{l>j}\sum_\alpha \Delta_{jl}^\alpha\,; \tag{28}$$

$$\Delta_T = \sum_{j=1}^r \mathbf{H}_j - \Delta\,. \tag{29}$$

2.5. **Induction procedure.** We collect in this subsection some information and some notations which will be used in all proofs which are based on induction on the rank of the cone. So, here, we assume that $r > 1$. We first define

$$\mathcal{A}^- = \mathrm{lin}\{L(c_1), ..., L(c_{r-1})\} \quad \text{and} \quad \mathcal{A}^+ = \mathrm{lin}\{L(c_r)\}\,,$$

and, in an analogous way,

$$\mathcal{N}_0^- = \bigoplus_{i<j\le r-1} \mathcal{N}_{ij} \quad \text{and} \quad \mathcal{N}_0^+ = \bigoplus_{j=1}^{r-1} \mathcal{N}_{jr}\,.$$

$\mathcal{N}_0^+$ is an ideal of $\mathcal{N}_0$, while $\mathcal{N}_0^-$ is a subalgebra. Clearly $\mathcal{A} = \mathcal{A}^- \oplus \mathcal{A}^+$ and $\mathcal{N}_0 = \mathcal{N}_0^- \oplus \mathcal{N}_0^+$.

Next, we define A^+, A^-, N_0^+, N_0^- as the exponentials of the corresponding Lie subalgebras. Then $S_0^- = N_0^- A^-$ is the solvable group corresponding to the cone Ω^-, determined by the frame $c_1, \cdots, c_{r-1}$, which is of rank $r-1$ as we wanted. The underlying space V^- for Ω^- is the subspace

$$V^- = \bigoplus_{1 \le i \le j < r} V_{ij} \, .$$

We will make an extensive use of the fact that

$$A = A^- A^+ \quad \text{and} \quad N_0 = N_0^- N_0^+$$

in the sense that the mappings

$$A^- \times A^+ \times \ni (a^-, a^+) \mapsto a^- a^+ \in A,$$

and

$$N_0^- \times N_0^+ \ni (y^-, y^+) \mapsto y^- y^+ \in A$$

are diffeomorphisms.

Now, let us define

$$\mathcal{Z}^- = \bigoplus_{j=1}^{r-1} \mathcal{Z}_j \, .$$

Then it is easily seen that $\mathcal{Z}^- \times \mathcal{Z}^-$ is mapped by Φ into the subspace $(V^-)^{\mathbf{C}}$. Moreover, $\Phi(\zeta, \zeta)$ belongs to Ω^- when $\zeta \in \mathcal{Z}^-$. So, we may define the Siegel domain $\mathcal{D}^-$ as

$$\mathcal{D}^- = \{(\zeta, z) \in \mathcal{Z}^- \times (V^-)^{\mathbf{C}} : \ \Im z - \Phi(\zeta, \zeta) \in \Omega^-\}.$$

Let us define $\mathcal{N}(\Phi)^- = \mathcal{Z}^- \oplus V^-$ and

$$\mathcal{N}(\Phi)^+ = \mathcal{Z}_r \oplus \bigoplus_{j \le r} V_{jr} \, .$$

Then, again, $\mathcal{N}(\Phi)^-$ is a subalgebra and $\mathcal{N}(\Phi)^+$ is an ideal of $\mathcal{N}(\Phi)$. We define $N(\Phi)^-$ and $N(\Phi)^+$ as their exponentials. Then $N(\Phi)$ is a semi-direct product

$$N(\Phi) = N(\Phi)^- N(\Phi)^+.$$

Clearly $N(\Phi)^-$ is the nilpotent step two "boundary" group corresponding to $\mathcal{D}^-$.

Finally, we want to decompose the group N. Let $N^- = N(\Phi)^-(N_0)^-$, and $N^+ = N(\Phi)^+(N_0)^+$. Then N is a semi-direct product $N = N^- N^+$. Moreover, the whole group S may be written as

$$S = N^- N^+ A^- A^+ = N^- A^- N^+ A^+.$$

Clearly, $S^- = N^- A^-$ is the solvable group acting simply transitively on $\mathcal{D}^-$.

3. Poisson integrals

The aim of this section is to prove the following partial result in view of the main theorem.

Theorem 3.1. *Let F be a bounded function on S annihilated by Δ and by $\mathbf{H}_j$, for $j = 1, ..., r$. Then*

$$\mathbf{H}_j^T F = 0 \quad \textit{for } j = 1, ..., r,$$

and

$$\mathcal{L}_j F = \sum_\alpha \mathcal{L}_j^\alpha F = 0 \quad \textit{for } j = 1, ..., r.$$

¿From the formulas of the previous section, it is clear that a bounded function on the domain $\mathcal{D}$ which is $\mathbf{H}$–harmonic satisfies the assumptions. Moreover, the first statement implies the second one. To prove the first one, we shall use the characterization of $\mathbf{H}$–harmonic functions in terms of Poisson-Szegö integrals on tube domains. More precisely, following Hua [Hu] and [FK], it is sufficient to prove that F, considered as a function on the tube domain $T_\Omega = V + i\Omega$, is the Poisson-Szegö integral of some bounded function on V. To do this, our main tool will be the possibility to write F, in different ways, as a Poisson integral related to some elliptic operators which annihilate F.

Let us first give some notations. From the last section, we know that every $g \in S$ may be written in a unique way as a product $(\zeta, x)na$, with $(\zeta, x) \in N(\Phi)$ and $n \in N_0$. We write π for the projection on $N(\Phi)$, given by $\pi(g) = (\zeta, x)$, and $\tilde{\pi}$ for the projection on N, given by $\tilde{\pi}(g) = (\zeta, x)n$.

We first recall previous results of two of the authors. Even if they are valid in the more general context of a semi-direct product, we give them in the present context. We consider elliptic operators which may be written as

$$L = \sum_{j=1}^r \alpha_j \mathcal{L}_j + \sum_{j=1}^r \beta_j \mathbf{H}_j^T \tag{30}$$

with α_j and β_j positive constants. Then L is a sum of square of vector fields plus a first order term $Z = Z(L)$, which is called the drift, and may be written as $Z = -\sum \gamma_j H_j$, with $\gamma_j = \alpha_j \chi + (2 + (j-1)d)\beta_j + d \sum_{k<j} \beta_k$. It follows from [DH] and [R] that the maximal boundary of L can be easily computed (it depends on the signs of $\lambda_j(Z) - \lambda_i(Z)$ for $i < j$). In particular, it is equal to $N(\Phi)$ if the sequence γ_j is a non–increasing sequence, and to N if it is an increasing sequence. Let us summarize the results that we shall use in the next proposition.

Proposition 3.2. *Let L be given by* (30)*, and γ_j as above.*
(*i*) *If L is such that γ_j is a non–increasing sequence, there is a unique positive, bounded, smooth function P_L on $N(\Phi)$ with $\int_{N(\Phi)} P_L(y)dy = 1$ such that bounded*

L-harmonic functions on S are in one-one correspondence with $L^\infty(N(\Phi))$ via the Poisson integral

$$F(s) = P_L f(s) = \int_{N(\phi)} f(\pi(sw)) P_L(w)\, dw. \tag{31}$$

(ii) If L is such that γ_j is an increasing sequence, there is a unique positive, bounded, smooth function $\tilde{P}_L$ on N with $\int_N \tilde{P}_L(y)dy = 1$ such that bounded L-harmonic functions on S are in one-one correspondence with $L^\infty(N)$ via the Poisson integral

$$F(s) = \tilde{P}_L f(s) = \int_N f(\tilde{\pi}(sy)) \tilde{P}_L(y)\, dy. \tag{32}$$

Moreover, for each given $\eta > 0$, we may choose the coefficients α_j and β_j so that (i) holds, and that

$$\int_{N(\Phi)} \tau(w)^\eta P_L(w)\, dw < \infty \tag{33}$$

where $\tau(w)$ is the distance of w from the unit element $e \in N(\Phi)$ with respect to any left–invariant Riemannian metric.

As we said, (*i*) and (*ii*) may be found in [DH] and [R]. The integrability condition may be found in [D], Theorem (3.10): a sufficient condition for (33) is that

$$\eta \sum 2\beta_j \lambda(H_j)^2 + \lambda(Z) < 0,$$

for all linear forms on $\mathcal{A}$ of the form $\lambda = \frac{\lambda_k + \lambda_p}{2}, \frac{\lambda_k}{2}$. The fact that this condition may be satisfied is elementary.

We have chosen to add a tilde every time that we are concerned with an operator whose maximal boundary is the whole group N. We then define P_L as an integral,

$$P_L(w) = \int_{N_0} \tilde{P}_L(wy)\, dy. \tag{34}$$

Let us remark that, in this case, the functions F which may be written as

$$F(s) = P_L f(s) = \int_{N(\phi)} f(\pi(sw)) P_L(w)\, dw, \tag{35}$$

with f a bounded function on $N(\Phi)$, constitute a proper subspace of the space of bounded functions which are annihilated by L. It is in particular the case for the Laplace-Beltrami operator, which is obtained for the values $\alpha = 2\beta = 1$, and has maximal boundary N.

The main step in this section is the next proposition. It has been proved in [DHP] for general homogeneous Siegel domains (non necessarily symmetric), and for more general operators. However, in the case of symmetric Siegel domains, which is the case under consideration, the proof may be simplified considerably. We include it for the reader's convenience.

Proposition 3.3. *Let F be a bounded function on S annihilated by Δ and by $\mathbf{H}_j$, for $j = 1, ..., r$. Then, there exists a bounded function f on $N(\Phi)$ such that F may be written as*

$$F(s) = P_\Delta f(s) = \int_{N(\phi)} f(\pi(sw)) P_\Delta(w)\, dw.$$

Proof. We already know that there exists some bounded function $\tilde{f}$ on N such that F may be written as $\tilde{P}_\Delta \tilde{f}$. Moreover, we may assume that $\tilde{f}$ is a continuous function and prove that, in this case, f is the restriction of $\tilde{f}$ on $N(\phi)$. Indeed, in the general case, we consider the sequence of functions F_m defined by

$$F_m(s) = \int_N \phi_m(n) F(n^{-1}s)\, dn = \tilde{P}_\Delta(\phi_m * \tilde{f})(s).$$

with ϕ_m an approximate identity which is compactly supported and of class $\mathcal{C}^\infty$. Clearly F_m tends to F pointwise. Let us assume that we have already proved the proposition for continuous functions. Then $F_m = P_\Delta(f_m)$. All the functions (f_m) are bounded by $\|f\|_{L^\infty}$, so we can extract a *–weak convergent sequence which converges to f. Then $P_\Delta(f_m)$ converges to $P_\Delta(f)$ pointwise. Hence $F = P_\Delta f$.

So, let $\tilde{f}$ be a bounded continuous function on N, and let $F = \tilde{P}_\Delta \tilde{f}$. To prove the proposition, we want to prove that, for each fixed $w \in N(\Phi)$, the function $y \mapsto \tilde{f}(wy)$ is constant on N_0. Indeed, assume that it is the case and denote by f the restriction of $\tilde{f}$ to $N(\Phi)$. Then, for $s = wya$ with $w \in N(\Phi)$, $n \in N_0$ and $a \in A$, we can write

$$\begin{aligned} F(wya) = \tilde{P}_\Delta \tilde{f}(wya) &= \int_{N(\Phi)N_0} \tilde{f}(wyavua^{-1}) \tilde{P}_\Delta(vu)\, dvdu, \\ &= \int_{N(\Phi)N_0} f(wyava^{-1}y^{-1}) \tilde{P}_\Delta(vu)\, dvdu, \\ &= P_\Delta f(wya). \end{aligned}$$

Let us finally remark that it is sufficient to prove that $y \mapsto \tilde{f}(y)$ is constant on N_0. Indeed, once we have proved this, for each $w \in N(\Phi)$ we have the same conclusion with F replaced by ${}_wF$, with ${}_wF(g) = F(wg) = \tilde{P}_\Delta({}_w\tilde{f})(g)$. Again ${}_w\tilde{f}(y) = \tilde{f}(wy)$ is constant, which we wanted to prove.

So, let us show that $y \mapsto \tilde{f}(y)$ is constant on N_0. Let us define

$$F_H(wya) = \int_{N_0} \tilde{f}(yaua^{-1}) \left(\int_{N(\Phi)} \tilde{P}_\Delta(vu)\, dv \right) du. \tag{36}$$

We claim that

$$F_H(g) = \lim_{t \to -\infty} F((\exp tH)g), \tag{37}$$

where H is the vector field $H = \sum_{j=1}^{r} H_j$. Indeed, writing

$$F(g) = \int_{N(\Phi)N_0} \tilde{f}(\tilde{\pi}(gvu))\tilde{P}_\Delta(vu)\, dvdu,$$

we have

$$F((\exp tH)wya) = F(w_t y_t a \exp tH) = \int_{N(\Phi)N_0} \tilde{f}(w_t y_t a v_t u_t a^{-1})\tilde{P}_\Delta(vu)\, dvdu.$$

For an element g of N we have used the notation

$$g_t = (\exp tH)g(\exp(-tH)).$$

It follows from (21) that $u_t = u$ for every $u \in N_0$, and that w_t tends to the unit element. This implies (37). We now claim that

$$F_H(wya) = F_H(ya) \tag{38}$$

$$\mathbf{H}_j F_H = 0 \quad \text{for } j = 1, ..., r \quad \text{and } \Delta F_H = 0 \tag{39}$$

$$H F_H = 0. \tag{40}$$

We have already proved (38). Then (39) follows from the fact that left and right translations commute. So, for every t, $F((\exp tH)g)$ is annihilated by the HW operators and the Laplacian. To see (40), we use again the fact that $u_t = u$ for every $u \in N_0$ and the formula (36) to obtain that

$$F_H(ya(\exp tH)) = F_H(ya).$$

Then (40) follows at once.

Finally, uniqueness in Proposition 33 implies that $y \mapsto \tilde{f}(y)$ if and only if F_H is constant. To prove that F_H is constant, we consider the function G defined on N_0A by $G(ya) = F_H(ya)$. Then clearly G is annihilated by all operators

$$D_j = -\chi H_j + 2(H_j^2 - H_j) + \sum_{i<j} \sum_{\alpha} ((Y_{ij}^{\alpha})^2 - H_j) + \sum_{j<k\leq r} \sum_{\alpha} ((Y_{jk}^{\alpha})^2 - H_k) \tag{41}$$

and by H. So, to complete the proof, it is sufficient to prove the following lemma.

Lemma 3.4. *Let G be a bounded function on N_0A which annihilated by the operators $H, D_1, ..., D_r$. Then G is constant.*

Proof. There is nothing to prove when $r = 1$. For $r = 2$, let us remark that G, which is annihilated by $H_1 + H_2$, is also annihilated by $H_1 - H_2$ since

$$(D_1 - D_2)G = -(\chi + 2)(H_1 - H_2)G = 0.$$

Therefore, $H_1 G = 0$ and $H_2 G = 0$ and so, G is a bounded function on the Abelian group $N_0 = \mathbf{R}^d$ annihilated by the Laplace operator. Hence G is constant.

Let us now consider $r > 2$. We assume that the lemma has been proved with r replaced by $r - 1$. We write G as a Poisson integral with respect to the operator

$$D = \sum \alpha_j D_j,$$

for which the drift $Z(D)$ is equal to $Z = -\sum \gamma_j H_j$, with $\gamma_j = (\chi+2+(j-1)d)\alpha_j + d\sum_{k<j}\alpha_k$. We first remark that we may choose the coefficients α_j so that the γ_j decrease for $j \geq 2$ (when $\chi + 2 > d$, one can even find a sequence α_j such that γ_j is decreasing, and conclude directly since every D–harmonic bounded function is constant). With this choice, the maximal boundary of D is the group $N_1 = \exp \mathcal{N}_1$, with

$$\mathcal{N}^1 = \oplus_{j>1}\mathcal{N}_{1j}.$$

We also define $N^1 = \exp \mathcal{N}^1$, with $\mathcal{N}^1 = \oplus_{1<i<j\leq r}\mathcal{N}_{ij}$. Every y in N_0 can be written in a unique way as $y_1 y'$, with $y_1 \in N_1$ and $y' \in N^1$. We define π_1 by $\pi_1(ya) = y_1$. Then, (see [DH]), there exists functions ν_D and ϕ such that

$$G(ya) = \int_{N_1} \phi(\pi_1(yau))\nu_D(u)\ du.$$

The function ϕ is bounded, and we can assume as before that it is continuous. Using notations of the subsection 2.5. on the induction procedure, we can also write $y \in N_0$ as y^+y^-. When y is in N_1, then y^+ belongs to $N_{1r} = \exp \mathcal{N}_{1r}$. We shall prove that $\phi(y)$ depends only on y^+. Again, to prove this, it is sufficient to prove that $\phi(y^-)$ is constant. Indeed, once we have proved this, we may apply it to ${}_{y^+}\phi$ (with ${}_{y^+}\phi(n) = \phi(y^+n)$), using the function ${}_{y^+}G$ in place of G.

In order to prove that $\phi(y^-)$ is constant, let us define, as before,

$$\begin{aligned}
G^{\#}(ya) &= \lim_{t\to-\infty} G((\exp tH_r)ya) \\
&= \lim_{t\to-\infty} \int_{N_1} \phi(\pi_1(y^-(y^+)_t au^-(u^+)_t a^{-1}))\nu_D(u^+u^-)\ du^+du^- \\
&= \int_{N_1} \phi(\pi_1(y^-au^-a^{-1}))\nu_D(u^+u^-)\ du^+du^-.
\end{aligned}$$

Here $u_t = (\exp tH_r)u(\exp(-tH_r))$. We have used the fact that $(y^-)_t = y^-$, and $(y^+)_t$ tends to the unit element. We have

$$G^{\#}(y^-y^+a^-a^+) = G^{\#}(y^-a^-) = G^{\#}(y^+y^-a^-a^+)$$

$$\begin{aligned}
D_j^{\#}G^{\#} &= 0, \quad \text{for } j = 1, ..., r-1, \\
H^{\#}G^{\#} &= 0,
\end{aligned}$$

where $H^{\#} = H_1 + H_2 + \cdots + H_{r-1}$, and, for $j = 1, ..., r-1$,

$$D_j^{\#} = 2H_j^2 - (\chi+2)H_j + \sum_{i<j}((Y_{ij}^{\alpha})^2 - H_j) + \sum_{j<k\leq r-1}\sum_{\alpha}((Y_{jk}^{\alpha})^2 - H_k).$$

From the induction hypothesis, we conclude that $G^{\#}$ is constant. So $\phi(y^-)$ is also constant. Hence $\phi(y) = \phi(y^+)$, and, using obvious notations, we conclude that G may in fact be written as

$$G(ya) = \int_{N_{1r}} \phi(\pi_{1r}(yau))\nu_D(u)\ du.$$

Since $\exp tH_j$ commutes with elements of N_{1r} for $j = 2, \cdots r-1$, we conclude that $H_jG = 0$. So $(H_1 + H_r)G = 0$. Moreover,

$$D_1G = \Big(2H_1^2 - (\chi+2)H_1 + \sum_\alpha((Y_{1r}^\alpha)^2 - H_r)\Big)G = 0$$

$$D_rG = \Big(2H_r^2 - (\chi+2+(r-2)d)H_r + \sum_\alpha((Y_{1r}^\alpha)^2 - H_r)\Big)G = 0,$$

which, as in the case $r = 2$, implies that G is constant. □

Once we have concluded for the lemma, we conclude for the proposition 3.3. □

Our next step is the following theorem.

Theorem 3.5. *Let f be a bounded function on $N(\Phi)$ and let $F = P_\Delta f$. Assume that*

$$\Delta_T F = 0.$$

Then

$$F((\zeta,x)ya) = \int_V f_\zeta(xyava^{-1}y^{-1})p(v)\ dv, \tag{42}$$

where $f_\zeta(x) = f(\zeta, x)$ and p is the Poisson - Szegö kernel for the tube domain $V + i\Omega$.

Proof. Using the same kind of proof as in the last proposition, we may assume that f is continuous. The maximal boundary for Δ_T considered as an operator on VS_0 is VN_0. Let $\tilde{p}$ be the corresponding kernel on VN_0. Then the function F_ζ, which is defined for $\zeta \in \mathcal{Z}$ fixed by $F((\zeta,x)s) = F_\zeta(xs)$, may be written as

$$F_\zeta(xya) = \int_{VN_0} g_\zeta(xyavua^{-1}))\tilde{p}(vu)\ dvdu, \tag{43}$$

where $v \in V$, $u \in N_0$.

We have also

$$F_\zeta(xya) = P_\Delta f((\zeta,x)ya) = \int_{N(\Phi)} f((\zeta,x)ya(\eta,v)a^{-1}y^{-1})P_\Delta(\eta,v)\ d\eta dv.$$

Let $a_t = \exp t(\sum_{j=1}^r jH_j)$. Then, on one hand,

$$\lim_{t\to-\infty} F_\zeta(xya_t) = g_\zeta(xy)$$

in *-weak topology on $L^\infty(VN_0)$ and on the other,

$$\lim_{t\to-\infty} F_\zeta(xya_t) = f(\zeta,x),$$

pointwise. Hence $g_\zeta(xy) = f(\zeta,x) = f_\zeta(x)$. Therefore,

$$\begin{aligned} F_\zeta(xya) &= \int_{VN_0} f_\zeta(xyava^{-1}aua^{-1})\tilde{p}(vu)\ dvdu, \\ &= \int_V f_\zeta(xyava^{-1}y^{-1})(\int_{N_0}\tilde{p}(vu)\ du)\ dv\,. \end{aligned}$$

It remains to prove that it is also equal to the right hand side of (42). But this last expression is a Δ_T–harmonic function (since the Poisson–Szgegö kernel is Δ_T–harmonic), with the same boundary values on VN_0. This proves (42). □

Proof of Theorem 3.1. Again, using Proposition 3.3, we may assume that $F = P_\Delta f$. Moreover, we may assume that f is continuous, using the same trick as in the proof of Proposition 3.3. So, it follows from Theorem 3.5 that F_ζ is a Poisson-Szegö integral on the tube domain. We know from [Hu] and [JK] that Poisson-Szegö integrals on symmetric tube domains are annihilated by Hua operators, i.e. in our situation F_ζ is annihilated by $\mathbf{H}_j^T$. This finishes the proof. □

4. The proof of pluriharmonicity

In this section we prove the following statement, which implies the main theorem:

Theorem 4.1. *Assume that*

$$\sup_{s\in S_0}\int_{N(\Phi)} |F((\zeta,x)s|^2\; d\zeta dx < \infty \tag{44}$$

and

$$\Delta F = \mathbf{H}_1 F = ... = \mathbf{H}_r F = 0. \tag{45}$$

Then F is pluriharmonic.

We first claim that the results of the last section on bounded functions apply to (H^2) growth conditions. Indeed, we have the following lemma. Here L is an elliptic operator as in (30),

$$L = \sum_{j=1}^r \alpha_j \mathcal{L}_j + \sum_{j=1}^r \beta_j \mathbf{H}_j^T\,,$$

with coefficients chosen so that it has maximal boundary $N(\Phi)$ and satisfies the integrability condition (33) for some η to be chosen later.

Lemma 4.2. *A function F which satisfies* (44) *and* (45) *may be written as a Poisson integral*

$$F(g) = \int_{N(\Phi)} f(\pi(gw))P_L(w)\; dw, \quad f\in L^2(N(\Phi)),\; g\in S. \tag{46}$$

Proof. We reduce to bounded functions by convolving F from the left. More precisely, let $\phi_n \in C_c^\infty(N(\Phi))$ be an approximate identity, and let

$$F_n(g) = \int_{N(\Phi)} \phi_n(w)F(w^{-1}g)\; dw.$$

Then F_n is bounded, and satisfies (45). So it follows from the last section that

$$F_n(g) = \int_{N(\Phi)} f_n(\pi(gw))P_L(w)\; dw,$$

for an $f_n \in L^\infty(N(\Phi))$. Moreover, f_n, which may be obtained as a *–weak limit in L^∞, when $t \to -\infty$, of $F_n(\cdot \exp tH)$ as well as a weak limit in $L^2(N(\Phi))$, is uniformly in L^2. Hence,

$$\|f_n\|_{L^2(N(\Phi))} \leq \sup_s \int_{N(\Phi)} |F(ws)|^2 \, dw$$

We may take for $f \in L^2(N(\Phi))$ the weak limit of a subsequence, and get (46). This concludes the proof of the lemma. $\square$

To prove Theorem 4.1, we may assume that $F = P_L f$ as above. Moreover, eventually convolving f in the group $N(\Phi)$ with a $\mathcal{C}^\infty$ compactly supported function as in the last section, we may assume that
Assumption on f: it may be written as $\phi * \tilde{f}$ where ϕ is a $\mathcal{C}^\infty$ compactly supported function.

At this point, our main tool will be harmonic analysis of the nilpotent group $N(\Phi)$. Once we have proved that the Fourier transform of f vanishes outside $\Omega \cup -\Omega$, one concludes easily like in [DHMP].

Let us first recall some basic facts about Fourier analysis on $N(\Phi)$, following [OV]. Let $(\ ,\ )$ be the Hermitian scalar product on $\mathcal{Z}$ for which the basis $e_{j\alpha}$, which was introduced in subsection 2.3, is orthonormal. It coincides with $4Q_j$ on each $\mathcal{Z}_j$, and these subspaces are pairwise orthogonal. For each $\lambda \in V$, let us define the Hermitian transformation $M_\lambda : \mathcal{Z} \to \mathcal{Z}$ by

$$4\langle \lambda, \Phi(\zeta, \omega)\rangle = (M_\lambda \zeta, \omega),\ \zeta, \omega \in \mathcal{Z}\,.$$

and consider the set

$$\Lambda = \{\lambda \in V : \det M_\lambda \neq 0\}$$

for which the above Hermitian form is non degenerate. Remark that it is in particular the case for $\lambda \in \Omega$ since we assumed that $\Phi(\zeta, \zeta)$ belongs to $\overline{\Omega} \setminus \{0\}$ for all $\zeta \neq 0$. The same is valid for $\lambda \in -\Omega$. So $\det M_\lambda$, which is a polynomial of λ, does not vanish identically, and Λ is an open set of full measure. It carries the Plancherel measure (see [OV]), given by

$$\rho(\lambda) d\lambda = |\det M_\lambda| d\lambda.$$

Let us describe the Fock representation associated to $\lambda \in \Lambda$. For every $\lambda \in \Lambda$ we define a complex structure $\mathcal{J}_\lambda$, which determines the representation space $\mathcal{H}_\lambda$. Let $|M_\lambda|$ be the positive Hermitian transformation such that $|M_\lambda|^2 = M_\lambda^2$. Then

$$\mathcal{J}_\lambda = i|M_\lambda|^{-1} M_\lambda.$$

If $\lambda \in \Omega$ then $\mathcal{J}_\lambda = iI = \mathcal{J}$ coincides with the ordinary complex structure in $\mathcal{Z}$. For general λ, the complex structure $\mathcal{J}_\lambda$ has a nice description in an appropriate basis. Namely, there is a λ-measurable choice of an $(\ ,\ )$ orthogonal basis $e_1^\lambda, ..., e_m^\lambda$ such that

$$H_\lambda(e_j^\lambda, e_k^\lambda) = \sigma_j \delta_{jk}$$

with $\sigma_j = \pm 1$ (depending on λ and locally constant). In the basis $e_1^\lambda, ..., e_m^\lambda, \mathcal{J}e_1^\lambda, ...,$ $\mathcal{J}e_m^\lambda$ of $\mathcal{Z}$ over $\mathbf{R}$ we have

$$\mathcal{J}_\lambda(e_j^\lambda) = \sigma_j(\mathcal{J}e_j^\lambda) \text{ and } \mathcal{J}_\lambda(\mathcal{J}e_j^\lambda) = -\sigma_j e_j^\lambda\,.$$

Let

$$B_\lambda = \Im H_\lambda\,.$$

A direct calculation shows that

$$B_\lambda(\mathcal{J}_\lambda e_j^\lambda, e_k^\lambda) = \delta_{jk}$$

and so

$$B_\lambda(\mathcal{J}_\lambda\zeta, \zeta) > 0 \text{ if } \zeta \neq 0.$$

We define $\mathcal{H}_\lambda$ as the set of all C^∞ functions F on $\mathcal{Z}$ which are holomorphic with respect to the complex structure $\mathcal{J}_\lambda$ and such that

$$F(\cdot)\rho(\lambda)^{\frac{1}{2}}e^{-\frac{\pi}{2}B_\lambda(\mathcal{J}_\lambda\cdot,\cdot)} \in L^2(\mathcal{Z}, dz).$$

Here dz is the Lebesgue measure related to the scalar product $(\cdot,\cdot)$ on $\mathcal{Z}$.

The space $\mathcal{H}_\lambda$ is a Hilbert space for the scalar product

$$(F_1, F_2)_\lambda = \int_{\mathcal{Z}} F_1(\zeta)\bar{F}_2(\zeta)e^{-\pi B_\lambda(\mathcal{J}_\lambda\zeta,\zeta)}\ \rho(\lambda)d\zeta.$$

The Fock representation U^λ, which is a unitary and irreducible representation on $\mathcal{H}_\lambda$, is given by

$$U^\lambda(\zeta, x)F(\omega) = e^{-2\pi i\langle\lambda,x\rangle-\frac{\pi}{2}|\zeta|^2+\pi\omega\bar{\zeta}}F(\omega-\zeta)\,, \tag{47}$$

with $\omega\bar{\zeta} = B_\lambda(\mathcal{J}_\lambda\omega,\zeta) + iB_\lambda(\omega,\zeta)$ and $|\zeta|^2 = \zeta\bar{\zeta}$. Then the Fourier transform of $f \in L^1(N(\Phi))$, which we note U_f^λ, is defined as the operator on $\mathcal{H}_\lambda$ given by

$$(U_f^\lambda F, G)_\lambda = \int_{N(\Phi)} f(\zeta, x)(U_{(\zeta,x)}^\lambda F, G)_\lambda dx.$$

If $f \in L^1(N(\Phi)) \cap L^2(N(\Phi))$, then the Plancherel theorem says that

$$\int_V \|U_f^\lambda\|_{HS}^2\ \rho(\lambda)d\lambda = \|f\|_{L^2(N(\Phi))}^2.$$

It follows that, for $f \in L^2(N(\Phi))$, U_f^λ is defined for almost every λ and is a Hilbert-Schmidt operator.

Now we write an orthonormal basis of $\mathcal{H}_\lambda$, which changes measurably with λ. For $\zeta \in \mathcal{Z}$, we note $\zeta_{j,\lambda}$ its coordinates in the basis e_j^λ, so that, in particular,

$$B_\lambda(\mathcal{J}_\lambda\zeta, \zeta) = \sum_j |\zeta_{j,\lambda}|^2\,.$$

Given a multi-index $\alpha = (\alpha_1, ..., \alpha_m)$, let

$$\xi_\alpha^\lambda = \frac{\pi^{\frac{|\alpha|}{2}}}{\sqrt{\alpha!}}\prod_j \zeta_{j,\lambda}^{\alpha_j\frac{(1+\sigma_j)}{2}}\bar{\zeta}_{j,\lambda}^{\alpha_j\frac{(1-\sigma_j)}{2}},$$

Then every ξ_α^λ is holomorphic with respect to the complex structure $\mathcal{J}_\lambda$ and the family $\{\xi_\alpha^\lambda\}$ forms a $(\ ,\ )_\lambda$ - orthonormal basis. Indeed, one may verify that

$$(\xi_\alpha^\lambda, \xi_\beta^\lambda)_\lambda = \frac{\pi^{\frac{|\alpha|+|\beta|}{2}}}{\sqrt{\alpha!\beta!}} \prod_j \int_{\mathbf{C}} u^{\alpha_j \frac{(1+\sigma_j)}{2}} \bar{u}^{\alpha_j \frac{(1-\sigma_j)}{2}} \bar{u}^{\beta_j \frac{(1+\sigma_j)}{2}} u^{\beta_j \frac{(1-\sigma_j)}{2}} e^{-\pi|u|^2}\, du.$$

We finally define, for $f \in L^2(N(\Phi)$ and almost every λ,

$$\hat{f}(\lambda, \alpha, \beta) = (U_f^\lambda \xi_\alpha^\lambda, \xi_\beta^\lambda)\,. \tag{48}$$

We may now give the main step of the proof.

Lemma 4.3. *Let $F = P_L f$ a function which satisfies the assumptions of Theorem* 4.1*, with L and $f \in L^2(N(\Phi))$ chosen as above. Then, for almost every λ and for all α, β, we have*

$$\hat{f}(\lambda, \alpha, \beta) = 0 \quad \textit{for} \ \ \lambda \notin \bar{\Omega} \cup -\bar{\Omega}. \tag{49}$$

Proof of Theorem 4.1. For the moment, we take the lemma for granted and finish the proof of Theorem 4.1. Let us first give some notations. For $s \in S_0$, we note F_s the function defined on $N(\Phi)$ by

$$F_s(\zeta, x) = F((\zeta, x)s) \tag{50}$$

and $\hat{F}(\lambda, \alpha, \beta, s)$ its Fourier transform. We claim that

$$\begin{aligned} \hat{F}(\lambda, \alpha, \beta, s) =& e^{-2\pi\langle \lambda, s\cdot e\rangle} (U_f^\lambda \xi_\alpha, \xi_\beta), \quad \text{for a.e } \lambda \in \bar{\Omega}, \\ =& e^{2\pi\langle \lambda, s\cdot e\rangle} (U_f^\lambda \xi_\alpha, \xi_\beta), \quad \text{for a.e } \lambda \in -\bar{\Omega}, \\ =& 0, \qquad\qquad\qquad\qquad\quad \text{for a.e } \lambda \notin \bar{\Omega} \cup -\bar{\Omega}. \end{aligned} \tag{51}$$

Indeed, we know from Theorem 3.5 that F may be written as a Poisson-Szegö integral, i.e.

$$F((\zeta, x)s) = \int_V f_\zeta(xsvs^{-1}) p(v)\, dv = \int_V f(\zeta, x-u) p_s(u)\, du\,,$$

with p_s defined by

$$p_s(u) = \det(s^{-1})\, p(s^{-1} \cdot u)\,.$$

Here the element s^{-1} is considered as acting on V. If $f \in L^1(N(\Phi)) \cap L^2(N(\Phi))$, then

$$\begin{aligned} (U_{F_s}^\lambda \xi_\alpha, \xi_\beta) =& \int_{N(\Phi)} \int_V f(\zeta, x-u) p_s(u) (U_{(\zeta,x)}^\lambda \xi_\alpha, \xi_\beta)\, du d\zeta dx \\ =& \int_V (U_f^\lambda U_{(0,u)}^\lambda \xi_\alpha, \xi_\beta) p_s(u)\, du \\ =& (U_f^\lambda \xi_\alpha, \xi_\beta) \int_V e^{-2\pi i \langle \lambda, u\rangle} p_s(u)\, du. \end{aligned}$$

These formulas are still valid for a general function $f \in L^2(N(\Phi))$: only use an approximation of f and the Plancherel theorem.

It remains to calculate the Fourier transform of p_s for $\lambda \in \bar{\Omega} \cup -\bar{\Omega}$. We shall do this for $\lambda \in \bar{\Omega}$. For $\lambda \in -\bar{\Omega}$ the proof is analogous. If $\lambda \in \bar{\Omega}$ we consider the bounded holomorphic function on $V + i\Omega$ given by

$$G(z) = e^{2\pi i\langle\lambda,z\rangle} = e^{2\pi i\langle\lambda,x+is\cdot e\rangle} = e^{2\pi i\langle\lambda,x\rangle - 2\pi\langle\lambda,s\cdot e\rangle}.$$

Then G is the Poisson integral of its boundary value, i.e.

$$G(z) = \int_V e^{2\pi i\langle\lambda,x-u\rangle} p_s(u)\ du.$$

Therefore,

$$G(is\cdot e) = e^{-2\pi\langle\lambda,s\cdot e\rangle} = \int_V e^{-2\pi i\langle\lambda,u\rangle} p_s(u)\ du\,.$$

Finally, for $\lambda \in \bar{\Omega}$, we have

$$(U^\lambda_{F_s}\xi_\alpha, \xi_\beta) = e^{-2\pi\langle\lambda,s\cdot e\rangle}(U^\lambda_f\xi_\alpha, \xi_\beta).$$

From (51), a direct computation (see [DHMP] for the details) shows that $\Delta_j F = 0$ for $j = 1, ..., r$. Moreover, we already know that $\mathcal{L}_j F = 0$. Then it follows from Theorem 3.1 in [DHMP] that F is the real part of an H^2 holomorphic function. $\square$

Proof of Lemma 4.3. It remains to prove the lemma. Let us remark that there is nothing to prove for $r = 1$. So the theorem is completely proved in this case. For $r > 1$, we can make the assumption that the theorem is valid for $r - 1$, and prove the lemma with this additional induction hypothesis.

We use again the notations of the subsection 2.5. for the induction procedure. An element $a \in A$ will be written as $a = a'a^+$, $a' \in A^-$, $a' \in A^+$. We call S'_0 the group N_0A^-, and S' the group NA^-. For $s \in S_0$, we may write $s = ya = ya'a^+ = s'a^+$.

We define a new function F' on S' by a limit process. More precisely, for $(\zeta, x)s' \in S'$, we define

$$F'((\zeta, x)s') = F'_{s'}(\zeta, x) = \lim_{t\to-\infty} F((\zeta, x)s' \exp tH_r). \tag{52}$$

Using the same arguments as before, as well as our assumptions on the boundary value f of F, one can see that this limit exists and is given by

$$F'_{s'}(\zeta, x) = \int_{N(\Phi)^-} f((\zeta, x)s'w^-(s')^{-1})P'_L(w^-)\ dw^-,$$

where

$$P'_L(w^-) = \int_{N(\Phi)^+} P_L(w^-w^+)\ dw^+.$$

We are now able to give a sketch of the proof. The function f may be seen as the boundary value of F'. So, we will consider the Fourier transform of $F'_{s'}$. Using the induction hypothesis for all functions ${}_{w^+}F'$, defined on S^- by ${}_{w^+}F'(s^-) = F(w^+s^-)$, we will show that ${}_{w^+}F'$ are pluriharmonic. This implies for their Fourier transforms

to satisfy a differential equation with initial data $f(\lambda, \alpha, \beta)$. Then smoothness of the Fourier transform will force this function to be zero for $\lambda \notin \bar{\Omega} \cup -\bar{\Omega}$.

Our main work will be to show the smoothness of Fourier transforms, and will ask for many technicalities.

Step 1: F' is a smooth function of arbitrary order on S'.

Proof. First, let W be a right-invariant differential operator on $N(\Phi)$. We know from the assumptions on f that Wf is well defined, and bounded. Therefore, we have

$$WF'_{s'}(\zeta, x) = \int_{N(\Phi)^-} Wf((\zeta, x)s'w^-(s')^{-1})P'_L(w^-)\ dw^-.$$

Moreover, partial derivatives of f grow at most polynomially. The action of s' is linear, hence there are constants $C(\alpha, K)$ and $M(\alpha)$ such that

$$|\partial^\alpha_{s'} f((\zeta, x)s'w^-(s')^{-1})| \le C(\alpha, K)(1 + \tau(w^-))^{M(\alpha)}$$

for $(\zeta, x)s'$ belonging to a compact set $K \subset S'$, with τ any left–invariant distance as in (33). Now we select η such that P_L integrates the right hand side above, to obtain

$$\int_{N(\Phi)^-} |\partial^\alpha_{s'} f((\zeta, x)s'w^-(s')^{-1})| P'_L(w^-)\ dw^- < \infty$$

which allows to differentiate F' with respect to s'. $\square$

Step 2: the function ${}_{w^+}F'$ satisfies the induction hypothesis on S^-.

Proof. We claim first that the assumption (44), with S^- in place of S, is satisfied for almost every w^+. Indeed, it is sufficient to prove that

$$\sup_{s' \in S'_0} \|F'_{s'}\|_{L^2(N(\Phi))} < \infty. \tag{53}$$

This follows from the fact that, for every $s' \in S'_0$, the function $F(\cdot s' \exp tH_r)$ has a weak limit in $L^2(N(\Phi))$ when t tends to $-\infty$. Indeed, for $\phi \in L^2(N(\Phi))$,

$$\begin{aligned} I &= \int_{N(\Phi)} (F(ws' \exp(t_1 H_r)) - F(ws' \exp(t_2 H_r)))\phi(w)\ dw \\ &= \int_{N(\Phi)} \int_{N(\Phi)} (f(wv_1) - f(wv_2) P_L(v)\phi(w)\ dv\ dw \end{aligned}$$

with $v_j = s' \exp(t_j H_r) v (s' \exp(t_j H_r))^{-1}$ for $j = 1, 2$. Integrating with respect to v over a compact set K and over its complement we get

$$I \le \sup_{v \in K} \|f(\cdot v_1) - f(\cdot v_2)\|_{L^2(N(\Phi))} \|\phi\|_{L^2(N(\Phi))} + 2\|f\|_{L^2(N(\Phi))} \|\phi\|_{L^2(N(\Phi))} \int_{K^c} P_L(v)\ dv,$$

which tends to zero when $t_1, t_2 \to -\infty$.

We now prove that the functions ${}_{w^+}F'$ satisfy the condition (45), again with S^- in place of S. Without loss of generality, we may assume that w^+ is the unit

element. Notice that the operators $\mathcal{L}_j, \Delta_j$ and Δ^α_{kj} have a perfect sense as left-invariant operators on S^- as far as indices are smaller than r . Let

$$(\mathbf{H}'_j)^T = 2\Delta_j + \sum_{k<j}\sum_\alpha \Delta^\alpha_{kj} + \sum_{j<k<r}\sum_\alpha \Delta^\alpha_{jk}.$$

Again $(\mathbf{H}'_j)^T$ may be considered as operators both on S and S^-. In the second case $(\mathbf{H}'_1)^T, ..., (\mathbf{H}'_{r-1})^T$ are HW operators for the tube $V^- + i\Omega^-$. We want to prove that, for $j = 1, \cdots, r-1$, we have

$$\mathcal{L}_j F' = (\mathbf{H}'_j)^T F' = 0.$$

Since for $i < j < r$, $\mathcal{L}_j$, Δ^α_{ij}, and Δ_j commute with A^+, we have, for $g' \in S'$,

$$\begin{aligned}\lim_{t\to-\infty} \mathcal{L}_j F(g' \exp tH_r) =& \mathcal{L}_j F'(g')\\ \lim_{t\to-\infty} \Delta_j F(g' \exp tH_r) =& \Delta_j F'(g')\\ \lim_{t\to-\infty} \Delta^\alpha_{ij} F(g' \exp tH_r) =& \Delta^\alpha_{ij} F'(g'),\end{aligned}$$

By hypothesis, F satisfies (45). So we conclude directly for $\mathcal{L}_j F'$, $j = 1, \cdots, r-1$. For $(\mathbf{H}'_j)^T F'$, we conclude also once we know that

$$\lim_{t\to-\infty} \Delta^\alpha_{jr} F(g' \exp tH_r) = 0 \tag{54}$$

Before doing it, we give a last definition. We note $\tilde{X}^\alpha_{jk}$, $\tilde{Y}^\alpha_{jk}$, and $\tilde{\mathcal{X}}^\alpha_j$, $\tilde{\mathcal{Y}}^\alpha_j$ the left-invariant vector fields on N which coincide, at the unit element of N, with the corresponding elements of the basis of $\mathcal{N}$ that we constructed in subsection 2.3. We define as well $\tilde{\mathcal{L}}_r = \sum_\alpha (\tilde{\mathcal{X}}^\alpha_r)^2 + (\tilde{\mathcal{Y}}^\alpha_r)^2$.

In the next computation, we identify an element a with a n-uple $(a_1, a_2, \cdots a_r)$, with $a_j > 0$, in such a way that a is the exponential of $\sum_j (\log a_j) H_j$. In particular, an element $a^+ \in A^+$ identifies with a scalar, which we note a_r for comprehension. With these notations, the previous limits are obtained for a_r tending to 0.

Then, it follows from the fact that $\mathcal{L}_r F = 0$ and a direct computation that

$$\chi \partial_{a_r} F(g' a_r) = \tilde{\mathcal{L}}_r F(g' a_r).$$

Moreover,

$$\Delta^\alpha_{jr} F(g' a_r) = a_r \Big(\sum_{j<r}\sum_\alpha a_j (\tilde{X}^\alpha_{jr})^2 + a_j^{-1} (\tilde{Y}^\alpha_{jr})^2 - \frac{1}{\chi}\tilde{\mathcal{L}}_r \Big) F(g' a_r) \to 0$$

when $a_r \to 0$. This finishes the proof of (54), as well as the claim of this step. Indeed, for almost w^+, the function ${}_{w^+}F'$ is pluriharmonic as a function on S'. It follows that $\Delta_1 F'$ vanishes identically. This is the main point which will be used later. □

step 3: $\hat{F}'(\lambda, \alpha, \beta, s')$ is a smooth function of s', for almost every λ and every α, β.

Proof. As before, $\hat{F}'(\lambda, \alpha, \beta, s')$ is the Fourier transform of the function $F'_{s'}$, defined on $N(\Phi)$ by $F'_{s'}(\zeta, x) = F((\zeta, x)s')$. We know from (53) that it is in $L^2(N(\Phi))$. Moreover, we can write the Fourier transform of F' in terms of the one of the Poisson kernel P'_L. Indeed, given $\lambda \in \Lambda$ and $G_1, G_2 \in \mathcal{H}^\lambda$, we define the bounded operator $U^\lambda_{P'_{s'}}$ by

$$(U^\lambda_{P'_{s'}} G_1, G_2)_\lambda = \int_{N(\Phi)^-} P'_L(w^-)(U^\lambda_{s'w^-(s')^{-1}} G_1, G_2)_\lambda dw^- \,,$$

For $f \in L^1 \cap L^2(N(\Phi))$, it follows directly from (52) that

$$\hat{F}'(\lambda, \alpha, \beta, s') = (U^\lambda_f U^\lambda_{P'_{s'}} \xi^\lambda_\alpha, \xi^\lambda_\beta)_\lambda \text{ for } a.e.\ \lambda \,.$$

For general $f \in L^2(N(\Phi))$, we use approximation in $L^2(N(\Phi))$ by integrable functions and the Plancherel theorem.

So, to prove the claim, it is sufficient to prove the smoothness $U^\lambda_{P'_{s'}}$ with respect to s'. This is given in the following lemma.

Lemma 4.4. *Assume that*

$$\int_{N(\Phi)} \tau(w)^{k+1} P_L(w)\ dw < \infty \,.$$

Then for every λ and every $G_1, G_2 \in \mathcal{H}^\lambda$ the function $s' \mapsto (U^\lambda_{P'_{s'}} G_1, G_2)$ is of class $\mathcal{C}^k$.

Proof. It follows from (47) that $(U^\lambda_{w^-} G_1, G_2)$ is a smooth function of w^- with bounded derivatives (see (6.41) in [OV] and (i), (ii) at the end of (4.1) in [OV]). Since the action $s'w^-(s')^{-1}$ of s' is linear,

$$|\partial^\alpha_{s'} (U^\lambda_{s'w^-(s')^{-1}} G_1, G_2)| \leq C(\alpha, K)(1 + \tau(w^-))^{|\alpha|}$$

for s' belonging to a compact set K. Hence the conclusion follows from the assumption on P_L. □

This is the end of the step 3. □

Step 4: Conclusion.

As we said before, we want to write the equation $\Delta_1 F' = 0$ on the Fourier transform side. We need a preliminary lemma, which will allow us to do it.

Lemma 4.5. *Let D be an element of the enveloping algebra of $\mathcal{N}(\Phi)^- \oplus (\mathcal{N}_0)^- \oplus \mathcal{A}^-$ considered as a left–invariant operator on S'. Then*

$$\sup_{s' \in S'_0} \int_{N(\Phi)} |DF'(ws')|^2\ dw < \infty \,.$$

Proof. Recall that ${}_{w^+}F'$ is pluriharmonic on S^-. So, by By the Harnack inequality, we have

$$|DF'(w^+w^-a')|^2 \leq C(D,B)\int_B |F'(w^+w^-a'g)|^2\, dg,$$

where B is a neighborhood of identity in S^-, and the constant $C(D,B)$ does not depend on w^+. We use the notation $g = u^-n^-b'$, with $u^- \in N(\Phi)^-$, $n^- \in N_0^-$ and $b' \in A^-$. Then, for $s' = ya' \in S'$, we write

$$\begin{aligned}\int_{N(\Phi)} |DF'(wya')|^2\, dw &= \int_{N(\Phi)} |DF'(y^+w^+w^-y^-a')|^2\, dw^-dw^+\\
&\leq C(D,B)\int_B\int_{N(\Phi)} |F'(y^+w^+w^-y^-a'u^-n^-b')|^2\, dw^-dw^+du^-dn^-db'\\
&= C(D,B)\int_B\int_{N(\Phi)} |F'(y^+w^+w^-y^-a'u^-n^-b')|^2\, dw^-dw^+du^-dn^-db'\\
&= C(D,B)\int_B\int_{N(\Phi)} |F'(wya'n^-b')|^2\, dwdu^-dn^-db',\end{aligned}$$

which is finite. In the above calculation we have used the fact that the action of y on $N(\Phi)$ is unipotent and we changed coordinates in $N(\Phi)^-$ in the following way

$$w^-y^-a'u^-(y^-a')^{-1} \to w^-,$$

which preserves the measure dw^-. $\square$

We will now prove that for almost every λ, we have

$$(-4\pi^2\langle\lambda, \mathrm{Ad}_{s'}X_1\rangle^2 + H_1^2 - H_1)\hat{F}'(\lambda,\alpha,\beta,s') = 0. \tag{55}$$

To do it, we first approximate F'. Namely, we take a sequence $\phi_n \in \mathcal{C}_c^\infty(N(\Phi))$ such that $0 \leq \phi_n \leq 1$, $\phi_n = 1$ on the ball of radius n, and such that, for every left-invariant differential operator D on $N(\Phi)$ of positive order, $D\phi_n \to 0$ uniformly when $n \to \infty$. We put

$$F_n'((\zeta,x)s') = \phi_n(\zeta,x)F'((\zeta,x)s').$$

A direct calculation shows that

$$\begin{aligned}\widehat{X_1^mF_n'}(\lambda,\alpha,\beta,s') =& (-2\pi i)^m\langle\lambda, \mathrm{Ad}_{s'}X_1\rangle \hat{F}_n'(\lambda,\alpha,\beta,s'),\\
\widehat{H_1^mF_n'}(\lambda,\alpha,\beta,s') =& H_1^m\hat{F}_n'(\lambda,\alpha,\beta,s').\end{aligned}$$

Then we let $n \to \infty$ and conclude for (55) using Lemma 4.5. Indeed, Lemma 4.5 implies that

$$\lim_{n\to\infty}\int_K\int_{N(\Phi)} |DF_n'((\zeta,x)s') - DF'((\zeta,x)s')|^2\, d\zeta dx ds' = 0$$

for $D = X_1^2, H_1$ or H_1^2 and any compact set K in S'.

Now, we prove that (55) and the smoothness of $\hat{F}'$ forces the Fourier transform of f to vanish outside $\overline{\Omega} \cup -\overline{\Omega}$. Let J be the set of $\lambda \in V$ such that all the principal minors of λ do not vanish (for the definition, see [FK], Proposition VI.3.10). Since J is dense in V, it is sufficient to consider $\lambda \in J$ such that $\lambda \notin \overline{\Omega} \cup -\overline{\Omega}$. Then there is $y_0 \in N_0$ such that $\lambda = Ad^*_{y_0}\lambda_0$ with $\lambda_0 = \sum_{k=1}^r b_k\, c_k$, $b_k \neq 0$ for $k = 1, ..., r$ (see e.g. [DHMP]). Substituting

$$h(\lambda, s') = \hat{F}'(\lambda, \alpha, \beta, y_0^{-1}s')$$

into (55), we obtain

$$\big(-4\pi^2\langle\lambda_0, Ad^*_{s'}\tilde{X}_1\rangle^2 + H_1^2 - H_1\big)h(\lambda, s') = 0,$$

or in coordinates $s' = ya'$, with a' identified with a $r-1$-uple,

$$\big(-4\pi^2\langle\lambda_0, Ad^*_{y}\tilde{X}_1\rangle^2 + \partial^2_{a_1}\big)h(\lambda, ya') = 0,$$

This, and boundedness of h with respect to s' imply that

$$h(\lambda, ya') = c(\lambda, y)\exp\big[-2\pi|\langle\lambda_0, \mathrm{Ad}_y X_1\rangle|a_1\big].$$

Letting $a' \to 0$ we get

$$\hat{f}(\lambda, \alpha, \beta) = c(\lambda, y).$$

Finally, by Lemma 1.27 of [DHMP]

$$\begin{aligned} h(\lambda, s') =& \hat{f}(\lambda, \alpha, \beta)\exp\big[-2\pi|\langle\lambda_0, \mathrm{Ad}_y X_1\rangle|a_1\big] \\ =& \hat{f}(\lambda, \alpha, \beta)\exp\Big[-2\pi\Big|b_1 + \frac{1}{2}\sum_{l>1} b_l|y^{1l}|^2\Big|a_1\Big] . \end{aligned} \tag{56}$$

Since for at least one $l > 1$ the sign of b_l is different from the one of b_1, (56) contradicts smoothness of $\hat{F}'$ with respect to y, unless $\hat{f}(\lambda, \alpha, \beta) = 0$. This concludes for the proof of the lemma. □

References

[BV] N.Berline, M.Vergne, Equations de Hua et noyau de Poisson, *Lecture Notes in Math.* **880** (1981) 1-51, Springer-Verlag.

[BBG] A. Bonami, J. Bruna, S.Grellier, On Hardy, BMO and Lipschitz spaces of invariant harmonic functions in the unit ball, *Proc.of the London Math. Soc.* **71** (1998), 665-696.

[D] E. Damek, Left-invariant degenerate elliptic operators on semidirect extensions of homogeneous groups, *Studia Math.* **89**(1988), 169-196.

[DH] E. Damek, A. Hulanicki, Boundaries for left-invariant subelliptic operators on semidirect products of nilpotent and abelian groups, *J.Reine Angew. Math.* **411** (1990), 1-38.

[DHMP] E. Damek, A. Hulanicki, D. Müller, M.Peloso, Pluriharmonic H^2 functions on symmetric irreducible Siegel domains, to appear in G.A.F.A.

[DHP] E. Damek, A. Hulanicki, R.Penney, Hua operators on bounded homogeneous domains in $\mathbf{C}^n$ and alternative reproducing kernels for holomorphic functions, *Journ. Funct. Anal.* **151** (1997), 77-120.

[FK] J. Faraut, A. Korányi,*Analysis on Symmetric Cones*, Clarendon Press, Oxford (1994).

[He] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York (1978).

[Hu] L.K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, Science Press, Peking (1958), Amer. Math. Soc. Transl., Math. Monograph 6 (1963).

[JK] K. Johnson, A. Korányi, The Hua operators on bounded symmetric domains of tube type, *Ann. of Math.* **111** (1980), 589-608.

[KN] Sh. Kobayashi, K. Nomizu, *Foundations of differential geometry. Vol. II.* Wiley and Sons, New York (1969).

[KW] A. Korányi, J. Wolf, *Realization of hermitian symmetric spaces as generalized half-planes,* Ann. of Math. **81** (1965), 265-288.

[K] J. Koszul, Sur la forme hermitienne canonique des espaces homogènes complexes *Canad.J. Math.* **7** (1955), 562-576.

[OV] R. D. Ogden, S. Vági, Harmonic analysis of a nilpotent group and function theory on Siegel domains of type *II Advances in Math.* **33** no.1 (1979), 31-92.

[R] A. Raugi, Fonctions harmoniques sur les groupes localement compact à base dénombrable, *Bull. Soc. Math. France, Mémoire* **54** (1977), 5-118.

[RV] H. Rossi, M. Vergne, Representations of Certain Solvable Lie groups on Hilbert Spaces of Holomorphic Functions and the Application to the Holomorphic Discrete Series of a Semisimple Lie Group, *Journ. Funct. Anal.* **13** (1973), 324-389.

[Ru] W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer Verlag, Berlin, 1980.

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